VIBRATION AND STABILITY OF PLATES USING BEAM-COLUMN ANALOGY

S.Z. AL-SARRAF and A.A. ALI

Building and Construction Engineering Department, University of Technology, Baghdad, Iraq

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This study deals with the problem of approximating plate behavior (both dynamic and static) by using equivalent grid-framework model. The emphasis, for plate analysis, is on the stability and vibration analysis. Numerical results are presented for several example problems, and they indicate that the adopted method is reasonably accurate. For vibration analysis of plates using beam-column analogy the percentage of error depends on mesh size.

Keywords: vibration, stability, plate, beam-column.

1. INTRODUCTION

The analysis of a continuum under in-plane or out-of-plane loads has been attempted in various forms: standard exact series solutions[1], the finite element technique[2,3], the finite strip method[4], the finite difference technique[5], and grillage analyses[6-8].

In this study a grillage method is adopted based on a formulation presented by Yettram and Husain[7] which can be most useful when dealing with some continuum problems where an exact series solution is difficult to obtain, e.g., plates with complex boundary conditions. The goal here is to introduce a simplified method depending on a grillage method taking in account the effect of membrane loads on plate behavior. Practically such cases may be seen in the slabs considering membrane action or in closed conduits subjected to in-plane hydrostatic pressures.

The main difficulty in designing a grid of orthogonally connected beams to simulate a continuum is due to the Poisson’s effect. This has a considerable effect on deflections and moment distributions in a continuum, whereas it has no effect on grids that consist of unidirectional beams. So, in order to simulate a continuum by an equivalent grid, the latter has to be designed in such a manner that its flexural behavior in all directions must be coupled. This coupling will be such that the grid deflection in any direction will produce curvatures in all other directions governed by the plate bending relationships.

In deriving the stiffness properties of the beam-column analog, the following assumptions are made:
1. The material is perfectly elastic.
2. The deflections are small relative to the thickness of the continuum.
3. The thickness of the continuum is small relative to its other dimensions.
4. The element used is plane and rectangular in shape.
5. The beams used for the analog are only fictitious beams used to construct the stiffness matrix of the element.
6. The moment intensities on the continuum element are constant along any edge.

The last assumption is true for infinitesimal elements, and the accuracy of the results will depend on the size of the mesh used. The model consists of side and diagonal beams as in Fig. 1b. The cross-sectional properties of the members are obtained by equating the rotations of the nodes of the grid with those of an element of equal size, when both are subjected to statically equivalent moments and torques. A rectangular grid model with five cross-sectional properties will define uniquely a rectangular element of a plate. These properties are chosen to be the flexural and torsional rigidities of the side beams and the flexural rigidity of the diagonals. A computer program is developed here using continuous mass method[9] and Wittrick-Williams method[10] for solving for eigenvalues[11].
2. EVALUATION OF THE CROSS-SECTIONAL PROPERTIES

By considering the dynamic stiffness matrix of each beam in the grid-framework model (Fig. 1), which is given in Appendix A for vibration-stability analysis and making appropriate substitutions of dynamic-stability functions given also in Appendix A, the governing matrix equation for the grid model will be:

$$\{F\} = [K]\{\delta\}$$

(1)

where $$\{F\}$$ and $$\{\delta\}$$ are force and displacement vectors and are given as

$$\{F\} = \begin{bmatrix} \mathcal{M}_b \mathcal{M}_f \mathcal{M}_s \mathcal{M}_i \end{bmatrix} \begin{bmatrix} \mathcal{M}_b \mathcal{M}_f \mathcal{M}_s \mathcal{M}_i \end{bmatrix}^T \{\phi\}$$

(2)

$$\{\delta\} = \begin{bmatrix} \dot{\theta}_1 \dot{\theta}_2 \dot{\theta}_3 \dot{\theta}_4 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \dot{\theta}_2 \dot{\theta}_3 \dot{\theta}_4 \end{bmatrix}^T \{\phi\}$$

(3)

In these matrices $$\theta_1$$, $$\theta_2$$, and $$\phi_i$$ are the rotations about the $$x$$- and $$y$$-directions and the transverse displacement; $$\mathcal{M}_b$$, $$\mathcal{M}_f$$, and $$\mathcal{M}_i$$ are the moments about the $$x$$- and $$y$$-directions and the transverse shearing force, for a node $$i = 1, 2, 3, 4$$ and $$[K]$$ is the $$(12 \times 12)$$ dynamic stiffness matrix for plate element where the nonzero elements of this matrix are

$$
k_{1,1} = k_{4,4} = k_{7,7} = k_{10,10} = \frac{GJ}{l} \left( \frac{\alpha_1 \cot \alpha_1}{k} + EI_1 F_2(\lambda_d) + EI_1 F_2(\lambda_d) \right)
$$


$$k_{1,2} = k_{4,5} = -k_{7,8} = k_{10,11} = -\frac{EI_1}{r^3 l} k F_2(\lambda_d)
$$

$$k_{1,3} = k_{4,6} = -k_{7,9} = -k_{10,12} = \frac{E}{l^2} \left( I_5 F_4(\lambda_d) + I_3 F_6(\lambda_d) \right)
$$

$$k_{2,2} = k_{5,5} = k_{8,8} = k_{11,11} = \frac{GJ}{l} \left( \frac{\alpha_1 \cot \alpha_1}{k} + EI_5 F_2(\lambda_d) + EI_5 F_2(\lambda_d) \right)
$$

$$k_{2,3} = -k_{5,6} = k_{8,9} = -k_{11,12} = -\frac{E}{l^2} \left( I_5 F_4(\lambda_d) + \frac{kF_4}{r^3} F_4(\lambda_d) \right)
$$

$$k_{2,5} = k_{8,11} = \frac{EI}{k l} F_1(\lambda_c)
$$

$$k_{2,6} = k_{8,12} = -\frac{EI}{k^2 l^2} F_3(\lambda_c)
$$

$$k_{2,8} = k_{5,11} = -\frac{GJ}{l} \alpha_1 \csc \alpha_1
$$

$$k_{2,11} = k_{5,8} = \frac{EI_d}{r^3 l} k^2 F_1(\lambda_d)
$$

$$k_{2,12} = -k_{5,9} = -\frac{EI_d}{r^3 l} k F_3(\lambda_d)
$$

$$k_{3,3} = k_{6,6} = k_{9,9} = k_{12,12} = \frac{E}{l^2} \left( I_5 F_6(\lambda_c) + I_3 F_6(\lambda_c) + \frac{I_d}{r^3} F_6(\lambda_d) \right)
$$

$$k_{3,5} = k_{9,11} = -\frac{EI}{k^2 l^2} F_4(\lambda_c)
$$

$$k_{3,6} = k_{9,12} = \frac{EI}{k l^3} F_5(\lambda_c)
$$

$$k_{3,7} = k_{6,10} = \frac{EI}{l^3} F_4(\lambda_s)
$$

$$k_{3,9} = k_{6,12} = \frac{EI}{l^3} F_5(\lambda_s)
$$

$$k_{3,10} = k_{6,7} = \frac{EI_d}{r^3 l^2} F_4(\lambda_d)
$$

$$k_{3,11} = -k_{6,8} = -\frac{EI_d}{r^3 l^2} k F_4(\lambda_d)
$$

$$k_{3,12} = k_{6,9} = \frac{EI_d}{r^3 l^2} F_5(\lambda_d)
$$

(4)

This is considered to be the dynamic stiffness matrix for plate element. For vibration with including the effect of in-plane forces, the frequency functions, $$F_i(\lambda), i = 1, 2, 6, \text{ above are replaced with stability-frequency functions } F_i(\alpha, b) \text{ as given in Appendix A.}$$ For stability analysis (12) the stability functions $$S_i(\beta), i = 1, 2, \ldots, 4, \text{ are used and the elements of dynamic stiffness matrix are modified by replacing } F_i(\lambda) \text{ to } F_i(\lambda) \text{ with } S_i(\beta) \text{ to } S_i(\beta) \text{ for } F_i(\lambda) \text{ with } S_i(\beta) \text{ and } F_i(\lambda) \text{ with } S_i(\beta).$$

For zero frequencies, the frequency functions $$F_i(\lambda), \text{ are reduced to the linear static case of plate-grid-framework that is given by Yettram and Husain [7], which are used in the derivation of properties of the constituting elements of grid model. For the grid to simulate the plate element, corresponding rotations must be equal for the two systems (Fig. 1); thus,

$$\theta_1 = \theta_6, \theta_2 = \theta_7, \theta_3 = \theta_8, \theta_4 = \theta_9 \text{ and } \theta_5 = \theta_{10} \text{ (5)}$$
\[
\theta_1 = \frac{kl M_1}{E \left( \frac{l^3}{12} \right)} \quad \theta_2 = \frac{\nu k l M_2}{E \left( \frac{l^3}{12} \right)} \quad \theta_3 = \frac{l M_2}{E \left( \frac{l^3}{12} \right)} \quad \theta_4 = \frac{\nu k l M_2}{E \left( \frac{l^3}{12} \right)} \quad \theta_5 = \frac{kl H(1 + \nu)}{E \left( \frac{l^3}{12} \right)}
\] (6)

in which \( l \) and \( kl \) are the side lengths of the element, \( t \)

is its thickness, \( \nu \) is Poisson's ratio and \( E \) is the elastic

modulus of the material.

The rotations \( \theta_6 \) to \( \theta_{10} \) are the corresponding

rotations of the grid-framework subjected to the same

moment and torque intensities and they are

\[
\begin{align*}
\theta_6 &= \frac{M_1 l^2 k}{2E} \left( \frac{r^3 I_s + I_d}{r^3 I_s I_c + I_d I_c + k^3 I_d I_s} \right), \\
\theta_7 &= \frac{M_1 l^2 k}{2E} \left( \frac{I_d}{r^3 I_s I_c + I_d I_c + k^3 I_d I_s} \right), \\
\theta_8 &= \frac{M_2 l^2 k^3}{2E} \left( \frac{r^3 I_c + k^3 I_d}{r^3 I_s I_c + I_d I_c + k^3 I_d I_s} \right), \\
\theta_9 &= \frac{M_2 l^2 k^3}{2E} \left( \frac{I_d}{r^3 I_s I_c + I_d I_c + k^3 I_d I_s} \right), \\
\theta_{10} &= \frac{H r^3 k l^2}{2E} \left[ r^3 \left( \frac{GJ_c}{E} \right) + 2 k l_d \right]
\end{align*}
\] (7)

As in Fig. 1b, the side beams of length \( l \) have

equal second moments of area \( I_s \) and equal torsion

factors \( GJ_s / E \), and the side beams of length \( kl \) have
equal second moments of area \( I_c \) and equal torsion

factors \( GJ_c / E \). The diagonals of length \( rl \) have second

moments of area \( I_d \) and no torsional stiffness. Of Eqs.

5 the second and fourth are identical and the first three

then provide, when expanded and solved, the second

moments of area of the grid members as

\[
\begin{align*}
I_s &= \frac{(k^2 - \nu^2) l}{2(k(1 - \nu^2))} \cdot I_c &= \frac{(1 - k^2 \nu^2) l}{2(1 - \nu^2)} \\
I_d &= \frac{\nu r^3 I_s}{2k(1 - \nu^2)}
\end{align*}
\] (8)

The last of Eqs. 5 gives the torsion factor

\[
\frac{GJ_c}{E} = \frac{(1 - 3 \nu) l}{2(l - \nu^2)}
\] (9)

Substituting \( \theta_{10} \) and transverse displacement at node 4,

\( w_4 \), where \( w_4 = -l \theta_{10} \) into the second of Eq. 1 yields

the remaining cross-sectional property and the grid
reduces to one consisting of side beams only, and of equal flexural and torsional rigidities.

\[
\frac{GJ_t}{E} = \frac{(1 - 3\nu)^2 l^4}{2(1 - \nu^2)} \frac{1}{12}
\]

(10)

when \( k = 1 \) and \( \nu = 0 \)

\[
I_x = I_y = \frac{h^3}{2} \frac{k}{12}, \quad I_z = 0, \quad \text{and}
\]

\[
\frac{GJ_t}{E} = \frac{GJ_s}{E} = \frac{1}{2} \frac{l^3}{12}
\]

(11)

3. PLATES UNDER COMBINED IN-PLANE AND OUT-OF-PLANE LOADS

The beam-column analog can also be used to determine the interaction between out-of-plane and in-plane loads through using the stability functions given in Appendix A. The critical load of the plate under any system of axial compressive loads can be determined by considering the effect of axial forces on the side beams only and neglecting the axial forces in the diagonal beams.

The in-plane distributed pressure is replaced by a statically equivalent system of concentrated loads at the edge nodes of the analog (Fig. 2). The effect of the in-plane compressive or tensile loads is introduced simply by substituting the values of the stability functions, corresponding to such loads, into the stiffness matrix of the corresponding member of the analog.

4. APPLICATION OF THE BEAM-COLUMN ANALOGY FOR THE DYNAMIC ANALYSIS

When considering the dynamic behavior there must be additional requirement that the mass per unit area of the grillage must be the same as that of the plate. In order to satisfy this condition, the total mass of the plate element must be the same as that of its equivalent beam-column analog element.

Here the diagonal beams will be assumed of zero mass. If each side member of the beam-analog element is assumed to be of constant mass per unit length, thus for an element of dimensions \( l \) and \( kl \), Fig. 3, the mass per unit length of the member of length \( l \) can be written as

\[
\mu = \frac{klt \rho}{4g}
\]

(12)

and that for the member with length \( kl \) as:

\[
\mu = \frac{lt \rho}{4g}
\]

(13)

in which \( l, \rho \) and \( g \) are the thickness, material specific weight of the plate and gravitational acceleration respectively. The values given in Eqs. 12 and 13 are those for boundary members and should be doubled for inside members.

If beside the harmonic excitation, the plate is subjected to in-plane static loads, these loads must be applied as statically equivalent lumped loads along the members of the beam-analog as explained previously.

The dynamic-stability functions, as given in Appendix A, are expressed in terms of the static and dynamic parameters \( \beta \) and \( \lambda \). If \( \beta \) and \( \lambda \) are calculated, the dynamic-stability functions can be directly determined. The dynamic behavior of the plate in the presence of in-plane static loads can thus be obtained by substituting these values of the dynamic-stability functions into the stiffness matrices of the members of the beam-analog.

5. NUMERICAL METHODS

The frequency of the discrete coordinate system may be given as follows

\[
[M] - \omega^2[K] = 0
\]

(14)

The formulation of Eq. 14 is an important mathematical problem known as a linear eigenvalue problem. There are many numerical methods dealing with eigenvalue and eigenvector problems. In the continuous mass method adopted here, the eigenvalue is of the type

\[
[K](\omega)D = 0
\]

(15)

where the dynamic stiffness matrix \([K(\omega)]\) is no longer a linear function of \( \omega^2 \) and Eq. 15 is known as a nonlinear eigenvalue problem. The dynamic stiffness matrix \([K(\omega)]\) has in general a transcendental dependence on \( \omega^2 \).

The solution of Eq. 15 needs methods different from those used to solve the linear eigenvalue problem given in Eq. 14. A powerful method presented by Wittrick and Williams is adopted here.

6. APPLICATIONS

In order to check the accuracy of the beam-column analogy when used for the analysis of vibration and stability of plates, several examples have been worked out numerically. The examples are chosen to represent different boundary conditions.
6.1 Example 1

To determine the interaction between out-of-plane and in-plane loads, a simply supported plate under the action of a uniform pressure, $N_x$, as well as an out-of-plane central load, $P$, is studied here, Fig. 3. The in-plane distributed pressure is replaced by equivalent system of concentrated loads at the edge nodes of the analog. Figure 4 shows the effect of the in-plane compressive load, represented by the non-dimensional factor, $\gamma = N_x / (4\pi^2 D/a^2)$, on the central deflections with comparison with Mohsin and Sadek's solution using their model of beam-analog. The critical value of this load is that at which $\delta \to \infty$ and, when extrapolated from Fig. 4, it is found by Mohsin and Sadek to be that corresponding to $\gamma = 0.97$.

A powerful numerical method presented by Wittrick and Williams is adopted here which is dealing with nonlinear eigenvalue problems. Using Wittrick-Williams method for solving the same case, it is found that $\gamma = 0.9996$ which is closer to the exact solution given by Timoshenko $\gamma_{exact} = 1.0$.

6.2 Example 2

Numerical solutions for rectangular plates of various boundary conditions are presented to illustrate the versatility and accuracy of the present method. In this example, all plates are solved using 4 × 4 mesh size. A number of rectangular plates of various homogeneous boundary conditions, mixed boundary conditions and point supports were analyzed.

The result for the lowest three modes are summarized in Table 1 and compared with those given by Fan and Cheung (1984) using spline finite strip method, analytical solution obtained by Leissa (1973) and Gorman (1981), finite element solution given by Rao (1975) and finite difference solution obtained by Cox (1955).

Results shown in Table 2 demonstrate that the grillage analogy gives good agreements even with a coarse mesh size of 4 × 4. For a finer mesh size of 8 × 8 more accurate results are obtained as for F-S-F-S case shown in Table 2.

6.3 Example 3

The fundamental frequency of a rectangular plate loaded by in-plane hydrostatic forces for a variety of aspect ratios, boundary conditions, and load magnitudes is treated in this example.

A comparison between the present study and various investigators are given in Table 3. As can be seen, there is a good agreement among the various methods. Table 3 shows that with increasing the in-plane compressive hydrostatic forces the fundamental frequency is reduced but with tensile forces the increasing of the forces results in increasing in the fundamental frequency parameter.

For a square plate, Fig. 6 shows a comparison of the dependencies of $\Phi^2$ where $\Phi = \omega a^2 \sqrt{\rho / g D}$ is non-dimensional frequency parameter on $K$ where $K = N a^2 / \pi^2 D$ is non-dimensional in-plane load parameter, for combinations of simply supported and clamped boundary conditions. It can be seen that a linear relationship is obtained between $\Phi^2$ and $K$. For the limiting condition of buckling, the critical in-plane loading parameter, $K_{cr}$, is shown in Fig. 7 for different boundary conditions and aspect ratios.
A presentation of the unloaded frequency parameter, $\Phi^*$, and the buckling load parameters, $K_{cr}$, for different boundary conditions and aspect ratios is given in Tables 4 to 7.

7. CONCLUDING REMARKS

Based on the results obtained in the present study, several conclusions may be drawn. These may be summarized as follows:

1. The plate element is represented by an equivalent grid-framework that deforms exactly as does the plate when both are subjected to statically equivalent loads. The method is applicable to a vast number of cases of plate flexure. The real value of Poisson's ratio is taken into account and thus the various boundary conditions can be considered without any difficulty. The degree of accuracy of the results obtained from beam-column analogy analysis depends mostly on the degree of fineness of the mesh used.

2. The beam-column analogy has been applied to typical examples of plate stability and vibration. The accuracy of the results as well as its rate of convergency are very satisfactory and compete well with other available methods even when relatively coarse meshes are used. Thus, the beam-column analogy can be considered as an analogy that can simulate both the static and dynamic behavior of plates, with or without in-plane forces, using the same routine program. It is believed that the accuracy of the results is mainly due to physically identifiable modeling of the plate elements by beam elements for which the deflection pattern is known and thus no arbitrary shape function is assumed.

3. Design curves for the fundamental natural frequencies have been presented for a rectangular plate with in-plane hydrostatic forces for different combinations of simply supported and clamped edges. It has been shown that the relationship between $\Phi^*$ and $K$ is, for most of practical purposes, linear. Buckling design curves also are obtained for different combinations of clamped and simply supported edges and values of aspect ratio.

Table 1. Natural frequencies, $f = (\omega/2\pi a^2)\sqrt{E/\rho/12(1-v^2)h^2}$ (hertz) of square plates (length = width) of various boundary conditions, Example 2.

<table>
<thead>
<tr>
<th>Mode sequence</th>
<th>Mode I</th>
<th>Mode 2</th>
<th>Mode 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fan and Cheung (1984)</td>
<td>23.65</td>
<td>51.71</td>
<td>58.67</td>
</tr>
<tr>
<td>Leissa (1973)</td>
<td>23.65</td>
<td>51.67</td>
<td>58.65</td>
</tr>
<tr>
<td>Present</td>
<td>23.46</td>
<td>50.46</td>
<td>57.06</td>
</tr>
<tr>
<td>Fan and Cheung (1984)</td>
<td>22.33</td>
<td>27.14</td>
<td>44.80</td>
</tr>
<tr>
<td>Leissa (1973)</td>
<td>22.27</td>
<td>26.53</td>
<td>43.66</td>
</tr>
<tr>
<td>Present</td>
<td>22.20</td>
<td>25.93</td>
<td>39.52</td>
</tr>
<tr>
<td>Fan and Cheung (1984)</td>
<td>9.80</td>
<td>17.02</td>
<td>37.90</td>
</tr>
<tr>
<td>Leissa (1973)</td>
<td>9.63</td>
<td>16.13</td>
<td>36.73</td>
</tr>
<tr>
<td>Present</td>
<td>9.64</td>
<td>15.69</td>
<td>33.23</td>
</tr>
<tr>
<td>Free</td>
<td>Simply supported</td>
<td>Clamped</td>
<td>Point support</td>
</tr>
</tbody>
</table>

| Finite strip* | Present | 22.82 | 49.25 | 55.32 |
| Finite strip | Present | 26.77 | 50.81 | 61.17 |
| Finite strip | Present | 28.66 | 61.06 | 62.48 |

Figure 7. Critical in-plane load parameter versus aspect ratio for various edge conditions.
Table 2. Comparison of the effect of mesh size on the accuracy of the results for Example 2.

<table>
<thead>
<tr>
<th>Mode sequence</th>
<th>j/4</th>
<th>1/2</th>
<th>3/4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fan and Cheung (15)</td>
<td>9.80</td>
<td>17.02</td>
<td>37.90</td>
</tr>
<tr>
<td>Leissa (16)</td>
<td>9.63</td>
<td>16.13</td>
<td>36.73</td>
</tr>
<tr>
<td>Present (4x4)</td>
<td>9.64</td>
<td>15.69</td>
<td>33.23</td>
</tr>
<tr>
<td>Present (8x8)</td>
<td>9.63</td>
<td>15.89</td>
<td>35.73</td>
</tr>
</tbody>
</table>

Table 3. Fundamental frequency parameter for a square clamped plate as a function of in-plane load parameter (Comparison of results from various investigators), Example 3.

<table>
<thead>
<tr>
<th>K</th>
<th>-15</th>
<th>-10</th>
<th>-5</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present (4x4)</td>
<td>68.006</td>
<td>59.354</td>
<td>48.985</td>
<td>35.236</td>
<td>31.681</td>
<td>27.619</td>
<td>22.773</td>
</tr>
<tr>
<td>Present (8x8)</td>
<td>68.524</td>
<td>59.870</td>
<td>49.528</td>
<td>35.914</td>
<td>32.427</td>
<td>28.480</td>
<td>23.831</td>
</tr>
<tr>
<td>Kiel and Han (15)</td>
<td>68.583</td>
<td>59.925</td>
<td>49.582</td>
<td>35.985</td>
<td>32.509</td>
<td>28.573</td>
<td>23.942</td>
</tr>
<tr>
<td>Dickinson (17)</td>
<td>-</td>
<td>61.13</td>
<td>-</td>
<td>36.13</td>
<td>-</td>
<td>28.62</td>
<td>-</td>
</tr>
<tr>
<td>Laura and Romanelli (20)</td>
<td>69.635</td>
<td>60.537</td>
<td>49.803</td>
<td>36.00</td>
<td>32.54</td>
<td>-</td>
<td>24.19</td>
</tr>
</tbody>
</table>

Table 4. Critical in-plane load parameter and unloaded fundamental frequency parameter for aspect ratio a/b = 1 for various edge conditions, Example (3).

<table>
<thead>
<tr>
<th>Edge condition</th>
<th>Present (4x4)</th>
<th>Kiel and Han (15)</th>
<th>Present (8x8)</th>
<th>Kiel and Han (15)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S-S-S-S</td>
<td>1.9813</td>
<td>2.0000</td>
<td>19.647</td>
<td>19.739</td>
</tr>
<tr>
<td>S-C-S-S</td>
<td>2.6213</td>
<td>2.6627</td>
<td>23.461</td>
<td>23.646</td>
</tr>
<tr>
<td>S-C-S-C</td>
<td>1.7276</td>
<td>3.2475</td>
<td>20.747</td>
<td>27.053</td>
</tr>
<tr>
<td>S-C-C-S</td>
<td>1.6318</td>
<td>4.3109</td>
<td>31.311</td>
<td>31.825</td>
</tr>
<tr>
<td>C-C-C-S</td>
<td>1.6318</td>
<td>4.3109</td>
<td>31.311</td>
<td>31.825</td>
</tr>
<tr>
<td>C-C-C-C</td>
<td>5.0669</td>
<td>5.3036</td>
<td>35.236</td>
<td>35.985</td>
</tr>
</tbody>
</table>

Table 5. Critical in-plane load parameter and unloaded fundamental frequency parameter for aspect ratio a/b = 1 for various edge conditions. Comparison of results using different mesh sizes, Example (3).

<table>
<thead>
<tr>
<th>Edge condition</th>
<th>Present (4x4)</th>
<th>Kiel and Han (15)</th>
<th>Present (8x8)</th>
<th>Kiel and Han (15)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S-S-S-S</td>
<td>1.9813</td>
<td>1.9977</td>
<td>2.0000</td>
<td>19.647</td>
</tr>
<tr>
<td>C-C-C-C</td>
<td>5.0669</td>
<td>5.2798</td>
<td>5.3036</td>
<td>35.236</td>
</tr>
</tbody>
</table>

Table 6. Critical in-plane load parameter and unloaded fundamental frequency parameter for aspect ratio a/b = 0.8 for various edge conditions using 8 x 8 mesh size, Example (3).

<table>
<thead>
<tr>
<th>Edge condition</th>
<th>Present (8x8)</th>
<th>Kiel and Han (15)</th>
<th>Present (8x8)</th>
<th>Kiel and Han (15)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S-S-S-S</td>
<td>1.6383</td>
<td>1.6400</td>
<td>16.177</td>
<td>16.186</td>
</tr>
<tr>
<td>S-C-C-S</td>
<td>2.6057</td>
<td>2.6129</td>
<td>21.308</td>
<td>21.338</td>
</tr>
<tr>
<td>C-C-C-C</td>
<td>4.4377</td>
<td>4.457</td>
<td>29.829</td>
<td>29.888</td>
</tr>
</tbody>
</table>

Table 7. Critical in-plane load parameter and unloaded fundamental frequency parameter for aspect ratio a/b = 0.6 for various edge conditions using 8 x 6 mesh size, Example (3).

<table>
<thead>
<tr>
<th>Edge condition</th>
<th>Present (8x6)</th>
<th>Kiel and Han (15)</th>
<th>Present (8x6)</th>
<th>Kiel and Han (15)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S-S-S-S</td>
<td>1.3585</td>
<td>1.3600</td>
<td>14.212</td>
<td>13.423</td>
</tr>
<tr>
<td>S-C-C-S</td>
<td>1.7650</td>
<td>1.7692</td>
<td>17.174</td>
<td>15.729</td>
</tr>
<tr>
<td>C-C-C-C</td>
<td>3.7752</td>
<td>3.7855</td>
<td>26.641</td>
<td>24.500</td>
</tr>
</tbody>
</table>

APPENDIX: General Stiffness Matrix

The stiffness matrix for any member of the analog shown in Fig. A.1 is
\[
\{f\} = [k]\{d\}
\]
where \(\{f\}\) and \(\{d\}\) are force and displacement vectors, and
\[
[k] = \begin{bmatrix}
\frac{GJ}{L}a_i cos\alpha & 0 & -\frac{GJ}{L}a_i cos\alpha & 0 & 0 \\
\frac{EI}{L}F_i & \frac{EI}{L}F_i & 0 & \frac{EI}{L}F_i & \frac{EI}{L}F_i \\
\frac{EI}{L}F_i & \frac{EI}{L}F_i & 0 & \frac{EI}{L}F_i & \frac{EI}{L}F_i \\
\frac{GJ}{L}a_i cos\alpha & 0 & -\frac{GJ}{L}a_i cos\alpha & 0 & 0 \\
\frac{EI}{L}F_i & \frac{EI}{L}F_i & 0 & \frac{EI}{L}F_i & \frac{EI}{L}F_i
\end{bmatrix}
\]
where
\[
\alpha_i = \omega L \left(\frac{\rho}{Gg}\right)^{\frac{1}{2}}
\]
in which \(\omega\) is the angular frequency, \(\rho\) is the specific weight of element, \(G\) is the shear modulus of the material and \(g\) is the gravitational acceleration.

The dynamic-stability functions, \(F_1\) to \(F_6\), for a beam subjected to harmonic loads, while under the action of an axial load, \(Q\), are given here briefly.

**Dynamic-Stability Functions**

1. **General case of dynamic excitation in the presence of axial static load.**

\[
F_1(a,d) = -\frac{(a^2 + d^2)a \sinh d - d \sin a}{f_1(a,d)}
\]

\[
F_2(a,d) = -\frac{(a^2 + d^2)d \cosh d \sin a - a \sinh d \cos a}{f_1(a,d)}
\]

\[
F_3(a,d) = -\frac{ad(a^2 + d^2) \cosh d - \cos a}{f_1(a,d)}
\]

\[
F_4(a,d) = \frac{ad(d^2 - a^2) \cosh d \cos a + 2ad^2 \sinh d \sin a}{f_1(a,d)}
\]

\[
F_5(a,d) = \frac{ad(a^2 + d^2) \sinh d + a \sin a}{f_1(a,d)}
\]

\[
F_6(a,d) = -\frac{ad(a^2 + d^2) \cosh d \sin a + a \sinh d \cos a}{f_1(a,d)}
\]

where

\[
a = \left[\frac{\beta^2}{2} + \left(\frac{\beta^2}{4} + \lambda^2\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}
\]

\[
d = \left[-\frac{\beta^2}{2} + \left(\frac{\beta^2}{4} + \lambda^2\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}
\]

\[
\beta = \frac{L}{\sqrt{EI}}
\]

\[
\lambda = \left(\frac{A\rho \omega_L^2}{EIg}\right)^{\frac{1}{2}}
\]

and

\[
f_1(a,d) = 2ad(d \cos \alpha \cos \beta - 1) + (a^2 - d^2) \sinh \alpha \sin a
\]

2. **Case of dynamic excitation with zero axial static load:**

This case can be directly obtained from the case number (1) by substituting \(\beta = 0\).

\[
F_1(\lambda) = -\lambda \frac{\sinh \lambda - \sin \lambda}{\cosh \lambda \cos \lambda - 1}
\]

\[
F_2(\lambda) = -\lambda \frac{\cosh \lambda \sin \lambda - \sinh \lambda \cos \lambda}{\cosh \lambda \cos \lambda - 1}
\]

\[
F_3(\lambda) = -\lambda^2 \frac{\cosh \lambda - \cos \lambda}{\cosh \lambda \cos \lambda - 1}
\]

\[
F_4(\lambda) = \lambda^2 \frac{\sinh \lambda \sin \lambda}{\cosh \lambda \cos \lambda - 1}
\]

\[
F_5(\lambda) = \lambda^2 \frac{\sinh \lambda + \sin \lambda}{\cosh \lambda \cos \lambda - 1}
\]

\[
F_6(\lambda) = -\lambda^3 \frac{\cosh \lambda \sin \lambda + \sinh \lambda \cos \lambda}{\cosh \lambda \cos \lambda - 1}
\]

3. **Case of static loading:**

a. **Axial compressive load:**

\[
F_1 = S_1(\beta) = \frac{\beta(\sin \beta - \beta)}{\beta \sin \beta + 2(\cos \beta - 1)}
\]

\[
F_2 = S_2(\beta) = -\frac{\beta(\sin \beta - \beta \cos \beta)}{\beta \sin \beta + 2(\cos \beta - 1)}
\]

\[
F_3 = S_3(\beta) = \frac{\beta^2 (\cos \beta - 1)}{\beta \sin \beta + 2(\cos \beta - 1)}
\]

\[
F_4 = -S_1(\beta)
\]

\[
F_5 = S_1(\beta) = \frac{\beta^3 \sin \beta}{\beta \sin \beta + 2(\cos \beta - 1)}
\]

\[
F_6 = -S_1(\beta)
\]
Vibration and Stability of Plates using Beam-Column Analogy

Figure A.1 Deformations $d_i$ and reactions $f_i$ in member coordinate axes system, $xyz$.

b. Axial tensile load:

\[ F_i = S_i(\beta) = \frac{\beta(\sinh \beta - \beta)}{\beta \sinh \beta - 2(\cosh \beta - 1)} \]  
\[ F_2 = S_2(\beta) = -\frac{\beta(\sinh \beta - \beta \cosh \beta)}{\beta \sinh \beta - 2(\cosh \beta - 1)} \]  
\[ F_3 = S_3(\beta) = \frac{\beta^2(\cosh \beta - 1)}{\beta \sinh \beta - 2(\cosh \beta - 1)} \]  
\[ F_4 = -S_4(\beta) \]  
\[ F_5 = S_5(\beta) = -\frac{\beta^2 \sinh \beta}{\beta \sinh \beta - 2(\cosh \beta - 1)} \]  
\[ F_6 = -S_6(\beta) \]

REFERENCES