SUPERSONIC TWO-DIMENSIONAL MINIMUM LENGTH NOZZLE DESIGN AT HIGH TEMPERATURE

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(Received November 2005 and accepted May 2006)

The goal of this research is to trace the profiles of the supersonic Plug Nozzle when this stagnation temperature is taken into account, lower than the threshold of dissociation of the molecules, by using the new formula of the Prandtl Meyer function, and to have for each exit Mach number, several nozzles shapes by changing the value of this temperature. A study on the error given by the PG model compared to our model at high temperature is presented. The comparison is made with the case of a calorically perfect gas aiming to give a limit of application of this model.

Keywords: Supersonic flow, minimum length nozzle, calorically imperfect gas, Prandtl Meyer function, stretching function, method of characteristics, relative error.

List of Symbols

- $M$: Mach number.
- $\theta$: flow angle deviation.
- $\nu$: value of the Prandtl Meyer function.
- $\mu$: Mach angle.
- $T$: temperature.
- $P$: static pressure.
- $\rho$: density.
- $a$: speed of sound.
- $\varepsilon$: tolerance of the calculation (desired precision).
- $A$: section area.
- $\gamma$: specific heat ratio.
- $C_p$: specific heat to constant pressure.
- $R$: constant of the gas.
- $H$: enthalpy.
- $L$: length of the nozzle.
- $F_x$: pressure force exerted on the wall of the nozzle.
- $C_{Mass}$: Coefficient of the Mass of the nozzle.
- $C_{Force}$: Coefficient of the pressure force.
- $l$: unit depth.
- $N$: number of regular right-running characteristics.
- $N_i$: number of inserted right-running characteristics.
- $\delta$: coefficient of the condensation function.
- $\xi$: right-running line of Mach.
- $\eta$: left-running line of Mach.
- $A$: interpolation coefficient of the pressure.
- $C_r$: interpolation coefficient of the ordinate $y$.
- $C_\theta$: interpolation coefficient of $\theta$.
- $C_T$: interpolation coefficient of Temperature.
- $HT$: abbreviation of the word High Temperature.
- $PG$: abbreviation of the word Perfect Gas.
- $MLN$: abbreviation of Minimum Length Nozzle.
- $C^+$: down -ward characteristics.
- $C^-$: up - ward characteristics.

Subscripts

- $i$: indice for average value between the points 1 and 3.
- $k$: indice for average value between the points 2 and 3.
- $j$: indice for the value of point 3.
- $0$: indice for chamber condition.
- $*$: indice for critical condition.
Traditionally, the supersonic nozzle is divided into two parts. The supersonic portion is independent of the upstream conditions of the sonic line. We can study the subsonic portion independently. The latter is used to give a sonic flow at the throat. In this class, we will study a type of nozzle giving a parallel and uniform flow at the exit section. It is named by Minimum Length Nozzle with centered expansion, which gives the minimal length compared to the other existing types. There are two categories for this nozzle according to the sonic line. If the sonic line is a straight line, the wall at the throat generates a centered and divergent expansion waves. The second category has a curved sonic line. In this case the flow inside the nozzle has not centered Mach lines. This type is named by Minimum Length Nozzle (MLN) with curved sonic line. Both types exist for two-dimensional and axisymmetric flows resulting in four possible configurations. The MLN with right sonic line \([1,2,7,8,10]\) studied the MLN axisymmetric with right sonic line. \([5]\) presents the first complete analysis of the two-dimensional curved sonic line MLN.

Figure 1 sketches a straight sonic line MLN. The flow between the throat OA and the downstream uniform flow consists of two regions. The area O O AB, named by region of Kernel, it is non-simple waves region. The transition region ABE is a simple wave region an its solution can be obtained analytically \([9]\).

The triangular region BSE is a region of a uniform flow having exit Mach number ME. In this case, the wall at the throat is tilted of an angle. The application of the MLN configuration with straight sonic line is limited for the gasodynamics \([16]\), where only the two-dimensional nozzle with straight sonic line is used.

However, the hypersonic blowers and the rocket motors, use the axisymmetric nozzle. The study in this research is limited for the 2D MLN configuration with straight sonic line. The works discussed before are for the case of a perfect gas to constant \(C_p\) and \(\gamma\). Those works are limited for low temperature stagnation, where we can go up to \(1000 K\) and for exit Mach numbers which do not exceed \(M_E=2.00\) approximately. In the real case, the specific heat \(C_p\) and the ratio \(\gamma\) vary with the temperature for the air and up to \(3550 K\), the table of the variation \([7]\).

The goal of this research is to add the effect of variation according to the temperature, of \(C_p\) and \(\gamma\) on the supersonic nozzle design, lower than the threshold of dissociation. The perfect gas in this case is named in more by gas calorically imperfect and thermally perfect or gas at High Temperature. In this case, the conservation equations remain unchanged except, the energy equation. The thermodynamic relations for HT model is presented in \([13]\), that of the Prandtl Meyer function \([12]\). The characteristics and compatibilities equations \((1-3)\) remain always valid in this form. Here we needs inserted the developed Prandtl Meyer function \([12]\) in the system to obtain our own valid mathematical model when the effect at high temperature is taken into account. Like results, the mathematical model developed in this research is a generalization of the characteristics equations model known in step badly of references \([9]\). Generally, the results in the gazodynamics are accepted for an error about 5%. A polynomial interpolation with the values of the table in order to find an analytical form for the function \(C_p(T)\) is applied. The presented mathematical relations are valid in the general case independently of the interpolation form and the substance, but our results will be presented by the choice of an interpolation of a polynomial of 9th degree \([13]\). The selected substance is the air. The comparison is made with the calorically PG model for goal to determine the limit of application of this model. A study on the error given by the PG model is presented in this case.

### 2. MATHEMATICAL FORMULATION

For a supersonic flow, non-rotational, 2D of a perfect gas, the method of characteristics gives the following equations \([1]\):

\[
\begin{align*}
\frac{dy}{d\xi} &= 0 \\
\frac{dy}{d\xi} &= \left(\frac{\gamma - 1}{\gamma}\right) \frac{dM^2}{d\xi} = \left(\frac{\gamma - 1}{\gamma}\right) \frac{dM^2}{d\xi} = \frac{1}{\gamma} \frac{dM^2}{d\xi}, \quad (1)
\end{align*}
\]

According to \(\xi (1-3)\):

\[
\begin{align*}
\frac{d(\gamma + \theta)}{d\xi} &= 0 \\
\frac{d\theta}{d\xi} &= \left(\frac{\gamma - 1}{\gamma}\right) \frac{dM^2}{d\xi} = \left(\frac{\gamma - 1}{\gamma}\right) \frac{dM^2}{d\xi} = \frac{1}{\gamma} \frac{dM^2}{d\xi}, \quad (1)
\end{align*}
\]
According to η (2–3):

\[
\begin{align*}
\frac{d(v-\theta)}{dx} &= 0 \\
\frac{dy}{dx} &= \tan(\theta + \mu)
\end{align*}
\]

Equations (1) and (2) are valid on \( C \) and \( C^+ \) respectively. In the real case, the characteristics are curved, and if the grid is fine so that the points are close one to the other, we can approaches the curve by a straight line and the work becomes on the lines of Mach named by \( \xi \) on \( C^- \) and by η on \( C^+ \). In Eqs. (1) and (2), the Prandtl Meyer function of our HT model is given by [12]:

\[
dv = F_c(T) = - \frac{C_p(T)}{2H(T)} \sqrt{M^2(T)-1} \ dt
\]

where

\[
M(T) = \frac{\sqrt{2H(T)}}{a(T)}
\]

\[
a(T) = \sqrt{\gamma(T) \frac{RT}{T}}
\]

\[
\mu(T) = \frac{C_p(T)}{C_p(T)-R}
\]

and \( R = 287.1029 \ J/(Kg \ K) \).

The interpolation coefficients of the polynomial \( C_p(T) \) as well as the function \( H(T) \) are presented in [14]. Replacing Eq. (3) in Eqs. (1) and (2) we obtain the mathematical model of the method of characteristics for High Temperature model.

As the function \( H(T) \) depends on the parameter \( T_0 \) [14], our mathematical model depends primarily on the stagnation temperature \( T_0 \). The developed mathematical model is a system of differential equations of four unknowns \( (x, y, T, \theta) \).

As \( C^- \) and \( C^+ \) are curved, the application of the method of characteristics obliges us to introduce a fine grid, in order to approximate each characteristic between two points by segments of straight lines. The properties \( (x, y, T, \theta, \rho, P) \) of the point 3 of Figure 2b can be given from those of the points 1 and 2 which connected it. We bring closer in this case the variation of the parameters \( y, \theta \) and \( T \) travelling the segments connecting the points 1 and 2 and the points 2 and 3 by the following expressions:

\[
\begin{align*}
\theta_{i3} &= \theta_i + (1-C_{\theta}) \ \theta_3 \quad i=1, 2 \\
T_{i3} &= T_i + (1-C_T) \ T_3 \quad i=1, 2 \\
y_{i3} &= y_i + (1-C_y) \ y_3 \quad i=1, 2
\end{align*}
\]

If the coefficients are equal to 0.5, we obtain the average value of the parameters.

2.1. Equations and Procedure for an Internal Point

The internal point flow field scheme is illustrated in Figure 3b. The integration of Eqs. (1) and (2) gives:

According to \( \xi \) (1 – 3):

\[
A_{13} \ (T_3 - T_1) + (\theta_3 - \theta_1) = 0
\]

\[
y_{13} - y_1 = C_{13} \ (x_{13} - x_1)
\]

According to η (2 – 3):

\[
A_{23} \ (T_3 - T_2) - (\theta_3 - \theta_2) = 0
\]

\[
y_{13} - y_2 = C_{23} \ (x_{13} - x_2)
\]

where

\[
A_{13} = - \frac{C_p(T_{i3})}{2H(T_{i3})} \sqrt{M_{13}^2-1} \quad i=1, 2
\]

\[
C_{13} = \tan(\theta_{13} - \mu_{13}) \quad , \quad C_{23} = \tan(\theta_{23} + \mu_{23})
\]

and

\[
\mu_{i3} = \arcsin \left( \frac{1}{M_{i3}} \right) \quad i=1, 2
\]

\[
M_{i3} = \sqrt{\frac{2H(T_{i3})}{a_{i3}}} \quad i=1, 2
\]

\[
a_{i3} = \sqrt{\frac{\gamma_{i3} \ R \ T_{i3}}{T_{i3}}} \quad i=1, 2
\]

\[
\gamma_{i3} = \frac{C_p(T_{i3})}{C_p(T_{i3})-R} \quad i=1, 2
\]

Equations (9), (10), (11) and (12) constitute a system of nonlinear algebraic equations with four unknowns \( (x_{i3}, y_{i3}, T_{i3}, \theta_{i3}) \). The successive iteration algorithm is written:

\[
x_{i3} = \frac{E_{i3} - E_1}{C_{i3} - C_{23}}
\]

\[
y_{i3} = E_1 + C_{13} \ x_{i3}
\]

\[
T_{i3} = \frac{D_{13} + D_{23}}{A_{i3} + A_{23}}
\]

\[
\theta_{i3} = D_1 - A_{i3} \ T_{i3}
\]

where:
The relations (18), (19), (20) and (21) give a system of equations of calculation by iterations for a supersonic 2D flow, permanent and irrotational for our HT model. The resolution of the system is done by the predictor corrector algorithm [15]. For the predictor algorithm, the initial values of \( y_{ib}, T_{i}, \) and \( \theta_{i} \) \((i=1, 2)\) for the iteration \( K=0 \) are given by:

\[
T_{i3}=T_{i1}, \quad \theta_{i3} = \theta_{i1}, \quad y_{i3} = y_{i1}
\]

\[
T_{23}=T_{21}, \quad \theta_{23} = \theta_{21}, \quad y_{23} = y_{21}
\]

Let us substitute Eqs. (24) and (25) in Eqs. (13), (14), (15), (16) and (17), and then in Eqs. (22) and (23), then replacing the results obtained in Eqs. (18), (19), (20) and (21) to obtain the initial values of \( (\Delta y^{0}, y^{0}, T^{0}, \theta^{0}) \) in a point 3.

For the corrector algorithm, values given by Eqs. (6), (7) and (8) are used and substituted in Eqs. (18), (19), (20) and (21) to obtain the new parameters values in point 3. The corrected values are \( (\Delta y^{1}, y^{1}, T^{1}, \theta^{1}) \).

The corrector algorithm will be repeated until arriving at the desired precision \( \varepsilon \). For \( K \) iterations, it is necessary to satisfy the following condition to ensure the convergence:

\[
\text{Max}\left[ \left| y_{3}^{K} - y_{3}^{K-1} \right|, \left| \theta_{3}^{K} - \theta_{3}^{K-1} \right|, \left| T_{3}^{K} - T_{3}^{K-1} \right| \right] < \varepsilon
\]

2.2. Equations for a Symmetry Axis Point

According to Figure 2c, point 3 is on the symmetry nozzle axis. Like \( y_{3}=0, \theta_{3}=0 \), the procedure is simplified and a line of Mach \( \zeta \) joining points 1 and 3 is employed. The valid equations on this line of Mach are to be used and the values of \( x_{3} \) and \( T_{3} \) are respectively obtained by the resolution of the algebraic equations (9) and (10). We obtain:

\[
x_{3} = x_{1} - \frac{y_{1}}{C_{13}}
\]

\[
T_{3} = T_{1} + \frac{\theta_{1}}{A_{13}}
\]

We can consider this point as an interior point, if we take the properties in point 2 by

\[
x_{2} = x_{1}, \quad y_{2} = -y_{1}, \quad \theta_{2} = -\theta_{1}, \quad T_{2} = T_{1}
\]

Once we determine the properties \((x, y, T \text{ and } \theta)\) in an unspecified point 3, we can determine the Mach number \( M3 \) by replacing \( T=T3 \) in Eq. (4). The density and the pressure ration can be determined by the following relations [15]:

\[
\left( \frac{\rho}{\rho_{0}} \right)_{3} = \text{Exp}\left( -\frac{T_{3}^{2}}{T_{0}^{2}} \frac{C_{p}(T)}{a^{2}(T)} \right)
\]

\[
\left( \frac{P}{P_{0}} \right)_{3} = \frac{T}{T_{0}} \left( \frac{\rho}{\rho_{0}} \right)_{3}
\]

3. CALCUlation PROCEDURE IN THE NOZZLE

3.1. Region of Kernel

The flow calculation in the Kernel region begins at point A of Figure 1. The initial expansion angle \( \theta^{*} \) is connected to \( M_{E} \) by the following relation [12]:

\[
\theta^{*} = \frac{v_{E}}{2}
\]

where [12],

\[
v_{E} = \int_{T_{E}}^{T} F_{E}(T) \ dT
\]

The calculation process of \( T_{e} \) and \( T_{E} \) [13]. Let us substitute the values of \( T_{e} \) and \( T_{E} \) in Eq. (31) to obtain the value of \( v_{E} \) and consequently, we can obtain \( \theta^{*} \) corresponding to \( M_{E} \). The integration procedure of Eq. (30) [13].

There is an infinity of Mach waves which result from the point A and which reflected on the symmetry axis. Numerical calculation obliges us to discretizing the flow zone expansion \( 0 \leq \theta \leq \theta^{*} \) in a finite number \( N \) of points. In total, we obtain \( N+1 \) \( C \) including. Between two regular successive characteristics, we choose:

\[
\Delta \theta = \frac{\Delta v}{N} \theta^{*}
\]

The relation (32) gives a uniform grid for the end \( C \) of the Kernel region and a broad space and non-uniform grid for the first \( C \). Consequently, the wall contour right after the throat will be badly presented. To correct this problem, we choose a grid refinement by insertion of additional \( C \) between the sonic line and the first regular \( C \). The distribution of the inserted \( C \) is given by introducing the following condensation:


\[ v_i = \left( \frac{I}{N_I} \right)^{\delta} \Delta v \quad i = 1, 2, 3, 4, ..., N_i \]  

(33)

The calculation procedure in the Kernel region is presented in Figure 3. In the first place, we determine the properties as in point 1 of Figure 3a. In this point, we have \( x_i = 0, y_i = y_A = y_M = 1 \) and \( \theta_i = \theta_A \) by using Eq. (33) if we choose a grid with condensation, and is equal to \( \theta_i = \theta_A \) by using Eq. (32) if the grid is without condensation. Temperature \( T_i \) must be determined by the resolution of the following equation [12]:

\[ \theta_1 = \frac{\int T}{T_i} F_r(T) \ dT \]  

(34)

The resolution procedure of Eq. (34) [12]. We proceed then to the determination of the properties in point 3 of Figure 3a by using the procedure of the symmetry axis point. We pass to second \( C \), and the calculation starts with the determination of the properties in point 1 of Figure 3b. In this point we always have \( x_i = 0 \) and \( y_i = y_M \), but \( \theta_i = \theta_A \) or \( \theta_i = 2 \Delta \theta \) according to the grid is with or without condensation. The determination of \( T_i \) is always done by resolution of Eq. (34) with the new value of \( \theta_i \). We consequently pass to the determination of properties as in point 3 of the Fig.4b by using the internal point procedure. Let us stop calculation on this \( C \) by the determination of the properties as in point 3 of Figure 3c by using the symmetry axis point procedure. Once that we arrive at the symmetry axis point, the \( C \) concerned is completely calculated. We pass to the third \( C \). Each characteristic starting from the third, contains 3 points types. The first, is point \( C \) confused with point \( A \), a point \( 3 \) on the symmetry axis as Figure 3f shows it, and the remainder points are of internal point type. Each type requires a different procedure as presented. The flow calculation in the Kernel region stops if calculation according to all \( N \) characteristic selected at the beginning is completed.

### 3.2. Wall Contour

The wall contour determination is done according to the process presented on Figure 4. The computing process has the recurrence continuation form, and calculation does not depend on the considered downstream results of the point. If we know the properties in a point, it is easy to determine those of the adjacent point and live towards that up to the exit section point.

The nozzle wall passes by point \( A \). In this point, we have \( x_i = 0, y_i = y_A = y_M = 1 \) and \( \theta_i = \theta_A \). The value of \( M' \) is equal to the Mach number in point \( A \) right after expansion and corresponding to the Mach number at the first point on last \( C \), see Figure 1.

The flow properties on each rising Mach line in the transition \( ABE \) area are constant, then the parameters \( (T, \theta, M, P, \gamma) \) on the wall points \( P_i (i = 1, 2, ..., N) \) are known and it remains us that to determine only the positions \((x, y)\) of each points. The Mach line rising in this region represents a portion of the isoMach curves.

\[
\begin{align*}
\theta_1 & = C_0 \ (\theta_{i+1} + (1-C_0) \ i) \\
\text{where,} \quad \alpha_{(i-1)} & = C_0 \ \theta_{i-1} + (1-C_0) \ \theta_i \\
\text{for} \quad i = 1, & \quad \frac{x_P}{y_P} = 0, \quad \frac{y_P}{y_P} = 1
\end{align*}
\]

When \( i = N \), we obtain the position of the last wall point which represents the exit section point. Then, in non-dimensional form, we have:

\[
\begin{align*}
x_E & = \frac{x_{PN}}{y_P} \quad \gamma_E = \frac{y_{PN}}{y_P}
\end{align*}
\]

The exit section ray corresponding to the discretization of \( N \) points is given, in non-dimensional form, by:

\[
\begin{align*}
\frac{AE}{x_p} & = y_E \ (\text{computed}) = \frac{y_{PN}}{y_P}
\end{align*}
\]

Figure 4. Process of determination of the wall points.

Figure 5. Presentation of calculation parameters of point \( P_i (i = 2, 3, 4, ..., N) \) of the wall.
The numerical results comparison obtained will be made between the ray of the exit section numerically calculated and the theoretical standardized critical sections ratio \((y*=1.0)\) presented by the following formula illustrated in \([14]\).

\[
\left(\frac{y*}{y}\right)_{\text{theoretical}} = AE = \exp\left(\int_{T_0}^{T_e} F_d(T) \, dT\right)
\]

where \(F_d(T)\) is given by \([11]\)

\[
F_d(T) = C_P(T) \left[\frac{1}{\sqrt{T^2(T)}} - \frac{1}{2H(T)}\right]
\]

It is very interesting to determine curves in the flow field of the nozzle, having even physical properties. These curves are called by iso values curves. Most interesting are isoMach and the iso directions curves. Let us note here, that the curves iso pressures, iso temperatures are they same the isoMachs curves.

### 3.3. IsoMachs Curves

Let \(M^\text{iso}\) the value of the Mach number that an internal whole of points in the nozzle must have, which we must determine their positions. We can have three cases, illustrated on Figure 6.

The properties \((x, y, M)\) in the points \(G\) and \(D\) are known and the problem is to determine the position \((x_P, y_P)\) of point \(P\) having \(M_P=M^\text{iso}\). The segment containing the point \(P\) must check the following condition:

\[
(M^\text{iso}-M_G) (M^\text{iso}-M_D) \leq 0
\]

(38)

To determine the position of point \(P\), we consider a linear variation \(M(S)=aS+b\) on the segment \(GD\). Since \(S=0\), we have \(M(S)=M_G\) and if \(S=S_{GP}\) (distance between points \(G\) and \(D\)), we have \(M(S)=M_D\). The relation \(M(S)=M_P=M^\text{iso}\) is checked when \(S=S_{GP}=S^\text{iso}\). Then, the distance between points \(G\) and \(P\) is given by:

\[
S_{GP} = S^\text{iso} = \frac{M^\text{iso}-M_G}{M_D-M_G} \sqrt{(x_G-x_D)^2 + (y_G-y_D)^2}
\]

(39)

The position \((x_P, y_P)\) can be determined by:

\[
x_P = x_G + S_{GP} \cos(\theta_{GP})
\]

(40)

\[
y_P = y_G + S_{GP} \sin(\theta_{GP})
\]

(41)

with:

\[
\theta_{GD} = \arctan\left(\frac{y_D-y_G}{x_D-x_G}\right)
\]

(42)

For the iso-direction curves we make the research of points on the basis of the flow angle deviation.

### 3.4. Mass of the Nozzle Structure

The wall segment numbering \((j)\) is illustrated in Figure 7. To manage to calculate the mass of the nozzle structure, we consider the two following assumptions:

- The shape of the wall between two successive points is approximated by a straight line.
- The divergent structure is made up of the same material of constant thickness \(t_M\) and density \(\rho_M\).

The calculation of the mass of structure dependent on the curvilinear length wall calculation, then, per unit of depth and in non-dimensional form, we obtain:

\[
\frac{\text{Mass}}{\rho_M \, t_M \, A*} = C_{\text{Mass}} = \frac{j=N-1}{\sum_{j=1}^{j=N-1} \left[\left(\frac{x_{j+1} - x_j}{y_* - y_j}\right)^2 + \left(\frac{y_{j+1} - y_j}{y_* - y_j}\right)^2\right]^{1/2}}^{1/2}
\]

(43)

### 3.5. Pressure Force Exercited on Divergent Wall

To calculate the pressure force exerted on the wall, it is still supposed that the pressure exerted on segment \((j)\) of Figure 7, named by \(P_{(j)}\), is approximated by:

\[
P_{(j)} = \sigma P_j + (1-\sigma) P_{j+1}
\]

(44)

For the applications, we take \(\sigma=0.5\).

The axial pressure force exerted on the panel \((j)\), per unit of depth, is given by:

\[
F_{x(j)} = P_{(j)} (y_{j+1}-y_j) \, l
\]

(46)

Then, the axial pressure force \(F_x\) exerted on the complete wall, is in non-dimensional form, given by:

\[
\frac{F_x}{P_0 \, A*} = C_{\text{Force}} = \sqrt{\sum_{j=1}^{j=N-1} \left(\frac{P_{(j)}}{P_0}\right) \left[\frac{y_{j+1} - y_j}{y_* - y_j}\right]^{1/2}}
\]

(47)

### 3.6. Error of the of Perfect Gas Model

The mathematical PG model is developed on the basis to consider \(C_p\) and \(\gamma\) as constants, which gives acceptable results with a certain error for weak stagnation temperature. According to this study, a difference between the results given by the PG model and our model will be presented. The error given by the PG model compared to our model can be calculated for each design parameter. Then for each couple \((T_0, M_0)\), the relative error \(\varepsilon\) can be evaluated by:

\[
\varepsilon_{\text{parameter}_{\%}} = \left|1 - \frac{\text{Parameter}_{\text{Perfect Gas}}}{\text{Parameter}_{\text{High Temperature}}}\right| \times 100
\]

(48)
The word parameter in Eq. (48) can represent all design parameters, in particular, the length, the mass, the pressure force and the ratio of the critical sections.

4. RESULTS AND COMMENTS

We have to prefer the presentation in each figure, three curves for the HT model corresponding to $T_0=1000 \text{ K}$, $2000 \text{ K}$ and $3000 \text{ K}$ including the PG model for $\gamma=1.402$.

In Figure 8, we have presented a characteristics grid with and without condensation effect. In this figure, we took an example for $T_0=1000 \text{ K}$ and $M_E=3.00$ which give $\theta^*=25.88$ degree. In this figure we made the tracing of 6 cases for different use.

Figure 8a presents a large grid without condensation, for $N=10$. We notice that the nozzle wall is badly presented in the vicinity of the throat, as well as a broad space between the sonic line and first regular $C$.

Figure 8b contains a grid with moderated refinement for $N=100$. We always notice, in spite of the number of point is raised enough, the wall shape in the vicinity of the throat is badly introduced still. But the bad presentation is less compared to the Figure 8a. Here the distance between the sonic line and the first regular $C$ is decreased a little but remains large compared to the other distances between successive $C$’s.

Figure 8c presents a large grid with additional inserted $C$ between the sonic line and the first regular $C$. The example presented is for $N=10$, $N_i=5$ and $\\delta=2$. The nozzle is presented by $15$ points. It is noticed that in spite of the characteristics number is weak, we will have a good wall presentation in the vicinity of the throat, or even on the horizontal axis in the vicinity of the throat. We can see the difference, let us look at the Figs. 8a and 8c of it.

On Figure 8d, we took the same $C$’s number and the same unserted $C$’s compared to Figure 8c, but we have changed the coefficient $\\delta$, which is equal to $7$ in this case. The nozzle will be presented with the same number of points as to Figure 8c except here, the wall in the vicinity of the throat will be presented in a very good way, which gives the utility of the condensation function.

On Figure 8e, we have presented a rather fine grid with $N=100$, $N_i=10$ and $\\delta=2$. In this figure, it is well presented at the throat.

In conclusion, it is not forcing if the number of nozzle points is large, we will have a good presentation. Here the factor of the wall points provision influences the obtained nozzle shape. In other words, we can design a nozzle with moderated number of points, if we use the condensation function and insertion of the additional characteristics between the sonic line and first regular $C$’s.

4.1. Effect of $T_0$ on the Nozzle Contour

Figure 9a takes the obtained nozzle form for $M_E=1.50$. This figure contains 4 curves, three for HT model for $T_0=1000 \text{ K}$, $2000 \text{ K}$ and $3000 \text{ K}$ including the case of PG model for $\gamma=1.402$, presented by curve 1. We can notice here a small difference between the four curves.

Let us increase the exit Mach number for the values $M_E=2.00$, $3.00$, $4.00$, $5.00$ and $6.00$, and for each Mach number, we make the tracing of the obtained nozzles shapes, presented respectively on cases (b-f) of Figs.9. Between the presented figures, we can say that if $T_0$ increases, the error between the PG model and HT model increased and becomes
considerable if \( T_0 > 1000K \), independently of the \( M_E \), or when \( M_E > 2.00 \) for any value of \( T_0 \). This limit can be found if we choose an error \( \varepsilon \) lower than 5%. Before the filling the Figure 9, we made a study on the discretization of the wall by increasing the number \( N \) and variation of \( N_l \) and \( \delta \) values, and each time we determine the dimensioning parameters and we calculate the relative error given by the non-dimensional exit ray by using Eq. (49) until satisfying an accuracy of \( 10^{-4} \).

\[
\varepsilon_{(y_E/y_*)} = \frac{1-(y_E/y_*)_{\text{computed}}/(y_E/y_*)_{\text{Theoretical}}}{100} \times 100
\]  

The calculated and theoretical \( y_E/y_* \) ratio are given by Eqs. (36) and (37). It was noticed that the length of Kernel, the length of nozzle, the mass of structure and the pressure force converge before the convergence of the exit section ray. This property is an advantage in order to control only the convergence of the exit ray. We still notice, that the parameters converge by a decreasing way.

4.2. Results on the Design Parameters

Figure 10 presents the variation of the initial expansion angle \( \theta^* \) versus \( M_E \). We notice the influence of \( T_0 \) on the value of \( \theta^* \) which goes influences the shape of the nozzle. Thus, if \( T_0 \) increases, more there is opening of the wall at the throat. The curves are almost confused until \( M_E = 2.00 \), then start to differentiate.

Figure 11 presents the variation of the length of Kernel region versus \( M_E \). Here, more the nozzle delivers a raised exit Mach number, more the length of the Kernel zone will be high. The goal to present this variation is that starting from this length, we can deduce the length of the nozzle directly without making the flow calculation in the zone of transition. Always we notice that the four curves merge with low Mach number until approximately \( M_E = 2.00 \). From this value, the four curves start to differentiate, and between curves 1 and 2 corresponds to PG model and HT model for \( T_0 = 1000 \) K, we can say that the theory of a perfect gas gives good results if this condition is checked. The length of nozzle can be given from the length of Kernel by the following relation:

\[
\frac{L}{y_*} = \frac{L_d}{y_*} + \frac{A_{E}}{A_{*}} \sqrt{\frac{M_E}{K} - 1}
\]  

The ratio of the sections in this relation must be calculated with Eq. (37). Figure 12 represents the variation of the length of nozzle versus \( M_E \). We always notice the influence of \( T_0 \) especially if this value exceeds 1000 K. Figure 13 represents the variation of the mass of the nozzle structure in non-dimensional form versus \( M_E \). If \( M_E < 2.00 \) approximately, the temperature \( T_0 \) does not present any influence on the mass of the structure.

Figure 9. Shapes of the nozzle giving at the exit \( M_E \). (a) : \( M_E = 1.50 \), (b) : \( M_E = 2.00 \), (c) : \( M_E = 3.00 \), (d) : \( M_E = 4.00 \), (e) : \( M_E = 5.00 \), (f) : \( M_E = 6.00 \).
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Figure 10. Variation of $\theta^*$ at the nozzle throat versus $M_e$.

Figure 11. Variation of the length of Kernel zone versus $M_e$.

Figure 12. Variation of the length of the nozzle versus $M_e$.

Figure 13. Variation of the mass of the structure versus $M_e$.

Figure 14. Variation of the pressure force versus $M_e$.

Figure 15. Variation of the relative error given by the design parameters of the PG model versus $M_e$. (a) Length, (b) Mass of structure, (c) Pressure force.
Applications. But if $T_0$ is raised, the error increases progressively and in this case, we can independently use the PG model at the temperature $T_0$ if $M_e$ does not exceed 2.00 with an error of approximately 10%.

When $M_e=1.0$, the error given by the PG model are given by the following values. Those values represent the intersection of the curves of the figure 15 with the error axis.

$$\varepsilon = \left( \frac{L}{y^*} \right)_{(M=1)} = \lim_{M \to 1} \left( \frac{L}{y^*} \right)_{HT} \times 100 = \frac{1}{0} \times 100 = 0$$

$$\varepsilon (C_{Mass}) (M=1) = \varepsilon (L/y^*) (M=1) = 0$$

$$\varepsilon (C_{Force}) (M=1) = \lim_{M \to 1} \left( \frac{C_{Force}}{C_{Force}} \right)_{HT} \times 100 = \frac{1}{0} \times 100 = 8.43\% \text{ when } T_0=3000K$$

$$= 1 - \frac{(P/R)_{GP}}{(P/R)_{HT}} \lim_{M \to 1} \frac{V_{GP}}{V_{HT}} \times 100 = 7.50\% \text{ when } T_0=2000K$$

$$= 4.73\% \text{ when } T_0=1000K$$

4.4. Various Results

Figure 16 represents the isoMach curves determined by the method of characteristics. It is noticed that the isoMach curves in the zone of transition are straight lines. The flow begins with entered 1D and ends in an exit 1D, but through the nozzle, the flow is 2D.

Figure 17 represents the iso directions curves. The points which has a null direction are the points of the throat, the horizontal axis and the all points of triangular uniform region. This figure also shows that the flow is two-dimensional.

In Figure 18 the forms of four nozzles having all same exit section are presented. Curves 1, 2 and 3 are for the HT model respectively for $T_0=1000K$, 2000K and 3000K. Curve 4 correspond to the PG model. The exit ray of the four curves corresponding to the case of PG model for $M_e=3.00$. We can show that they do not deliver the same Mach number $M_e$ starting from the relation (4). The goal to present this figure is that if we consider the nozzle dimensioned on the basis and the assumptions of a PG model for the aeronautical applications, we can notice the degradation of the performances, in particular the exit Mach number and the others parameters, considering the nozzles have almost same size and form, except a small difference in length. The flow in this difference in length is almost uniform. The shape of the nozzle used does not change except the thermodynamic behavior of the air with the temperature.

For example, if the nozzle delivers $M_e=3.00$ on the assumption of PG model, it will deliver, on the consideration of HT model, $M_e(HT)=2.94$, $M_e(HT)=2.84$ and $M_e(HT)=2.81$ respectively if $T_0=1000 K$, $T_0=2000 K$ and $T_0=3000 K$. Between the case of PG model and HT model, when $T_0=2000 K$, the fall of $M_e$ is equal approximately to 5.2%.
5. CONCLUSIONS

From this study, we can quote the following points:

- If we accept an error lower than 5% which is generally the case for the aerodynamic applications, we can study a supersonic flow by using the PG relations if \( T_0 \) is lower than 1000 K for any value of Mach number, or when the Mach number is lower to 2.00 for any stagnation temperature value up to approximately 3000K.

- The PG model is represented by a simple and explicit relations, and it doesn’t need high time to make calculation, which is not the case for our model, where it is represented by the resolution of a nonlinear algebraic equation and integration of two complex analytical functions requiring a calculation, and high time and data-processing programming.

- The basic variable for our model is the temperature, and for the PG model is the Mach number; because of a nonlinear implicit equation connecting the parameters \( T \) and \( M \).

- The relations presented in this study are valid for any interpolation type chosen for the function \( C_p(T) \). The essential one is that the selected interpolation gives an acceptable small error.

- We can choose another substance instead of the air. The relations remain valid, except here, it is necessary to have the table of \( C_p \) and \( \gamma \) variation according to the temperature and to make a suitable interpolation.

- We can obtain the relations of a perfect gas starting from the relations of our HT model by cancelling all the constants of interpolation of function \( C_p(T) \) except the first. In this case, the PG model becomes a particular case of our HT model.

- We can use the Prandtl Meyer function of our model to solve problems of the external flows in a hot medium. In particular, the flow around a supersonic pointed airfoil.

- The convergence of the relation (26) requires more iterations for the HT model compared to the PG model for same precision \( \varepsilon \). The difference between the iteration count that exist between the two models is approximately 40 %, which influences over the computing time per computer.

- For same precision \( \varepsilon \), the number of characteristics \( N \) necessary for HT model is higher than the number of characteristics for the PG model. This difference is due to the difference in length which requires more discretization.

- The ratio of the critical sections presented by the relation (37) can be used like a source of comparison for validating the numerical results of various supersonic nozzles dimensioned, giving an uniform and parallel flow at the exit section by the method of characteristics, and the Prandtl Meyer function[11,13,15].

- The design method and calculation in the nozzle is the same one between the two models except the equations of calculation changes in particular the Prandtl Meyer function.

Acknowledgment

The authors would like to thank the authorities of the university SAAD Dahleb of Blida and the Department of Aeronautics for the financial support granted for the completion of this research, without forgetting to thank Mr Djamel ZEBBICHE and Mth Fettoum MEBREK for time that they gave to me for the seizure of this manuscript.

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